Solutions

Math 1A Worksheet: Implicit Differentiation

Name: ____________________

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1. Suppose $x^2 + xy + y^2 = 3$. Find y' and y''. i) $2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$

ii.) $2 + \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{(x+2y)}\left(2 + 2\left(-\frac{(2x+y)}{x+2y}\right) + \left(-\frac{(2x+y)}{x+2y}\right)^2\right)$

2. Find the derivative of x^x. (Hint: let $y = x^x$, then $\ln(y) = x\ln(x)$. Use implicit differentiation!) What about the derivative of x^{x^x}?

$\ln(y) = x\ln(x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{x}{x} + \ln(x) \Rightarrow \frac{dy}{dx} = x^x(1+\ln(x))$

For x^{x^x}, $\ln(y) = x^x\ln(x) \Rightarrow \frac{1}{y}\frac{dy}{dx} = (x^x(1+\ln(x)))\ln(x) + x^{x-1}$

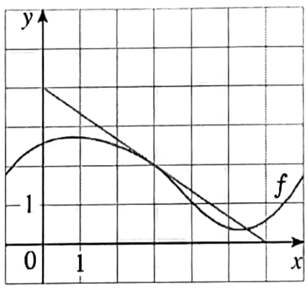
$\Rightarrow \frac{dy}{dx} = x^{x^x}(x^{x-1} + x^x(1+\ln(x))\ln(x))$

3. If $g(x) = \sqrt{f(x)}$, where the graph of f is shown. Find $g'(3)$.

$f'(3) = -2/3$; $f(3) = 2$

$g'(3) = \frac{1}{2\sqrt{f(3)}} f'(3)$

$= \frac{1}{2\sqrt{2}} \cdot -2/3 = -\frac{1}{3\sqrt{2}}$



4. The Bessel function of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

   a) Find $J'(0)$. $0 \cdot J''(0) + J'(0) + 0 \cdot 1 = 0 \Rightarrow J'(0) = 0$

   b) Use implicit differentiation to find $J''(0)$.

$y'' + xy''' + y'' + xy' + y = 0$

$\Rightarrow 2J''(0) + 0 \cdot J'''(0) + 0 \cdot J'(0) + 1 = 0$

$J''(0) = -1/2$

5. For each of the following equation, explain what geometric figure is represented by the equation, then find the slope of the tangent line at a point (x_0, y_0):

$$y = x^2$$

Parabola,

$2x_0$

$$x^2 + y^2 = 4.$$

Circle . $2x + 2y\frac{dy}{dx} = 0$

$-x_0/y_0$ for $y_0 \neq 0$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Ellipse

$-\dfrac{9x_0}{4y_0}$

$$xy = 1.$$

Hyperbola

$-1/(x_0)^2$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

Hyperbola

$9x_0/4y_0$

$$y^2 = x^2(x+1) \quad \text{(nodal elliptic curve)}$$

$2y \cdot \frac{dy}{dx} = 2x(x+1) + x^2 = 3x^2 + 2x$

$\frac{dy}{dx} = \dfrac{3x^2+2x}{x^2(x+1)}$

6. Let $f(x) = \cos(2x)$, and write $f^{(n)}(x)$ as the function obtained by differentiating f repeatedly for n times. Find $f^{(n)}(x)$ for $n = 1, 2, 3, 4$. What should $f^{(100)}(x)$ be?

$f^{(n)}(x) = 2^n (-1)^{\frac{n(n+1)}{2}} \cdot g_n(x)$ for $g_n(x) = \begin{cases} \cos(2x) & n \text{ even} \\ \sin(2x) & n \text{ odd} \end{cases}$

7. Suppose θ is measured in degrees instead of radians, use the Chain Rule to show that

$$\frac{d}{d\theta}(\sin(\theta)) = \frac{\pi}{180}\cos(\theta).$$

Derive a similar formula for the derivative of $\cos(\theta)$ and $\tan(\theta)$.

$\theta = \frac{180}{\pi} \cdot \varphi$ for φ in radians.

Chain Rule: $\frac{d}{d\theta}\sin(\varphi) = \frac{d\varphi}{d\theta} \cdot \frac{d}{d\varphi}\sin(\varphi)$

$= \frac{\pi}{180}\cos(\varphi) = \frac{\pi}{180}\cos(\theta)$

8. Use implicit differentiation to derive a formula for the derivative of inverse cosine function $y = \cos^{-1}(x)$.

$\cos(y) = x \Rightarrow -\sin(y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \dfrac{-1}{\sin(\cos^{-1}(x))}$

$= \dfrac{-1}{\sqrt{1-x^2}}$