GSI

Math 1A Worksheet: Definition of Limits

Name: ________________

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1. Let $f(x) = x^2$. Find

$$\lim_{x \to 5} f(x). \quad = 25$$

Then explain what this limit means using rigorous language. (Use the phrase "$|x^2 - 25| < \varepsilon$" instead of "x^2 is very closed to 25.")

$|x^2 - 25| = |(x+5)(x-5)| \le \varepsilon$ for $|x-5| < \varepsilon/12$, $\varepsilon < 12$

2. Let $f(x) = 2x + 4$. Find a positive number $\delta > 0$, such that for every x is in the interval $(3 - \delta, 3) \cup (3, 3 + \delta)$, the following inequality holds:

$$|f(x) - 10| < 1.$$

$\delta = 1/2$

3. Prove the following limits hold, using the ϵ-δ definition of limits.

$$\lim_{x \to 2} x = 2.$$

Obv.

$$\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}.$$

$|\frac{1}{x} - \frac{1}{2}| = |\frac{2-x}{2x}|$

$|x-2| < \varepsilon$, $\varepsilon < 1$

$$\lim_{x \to 0} \frac{1}{x^4} = \infty.$$

$\delta < 1/\sqrt[4]{M} \Rightarrow 1/x^4 > M$

$|x| < \delta$

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

For $|x| > N$, $|\frac{1}{x}| < 1/N$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = 2x.$$

$= \lim_{h \to 0} 2x + h$

$|h| < \varepsilon$

$\Rightarrow |2x + h - 2x| < \varepsilon$

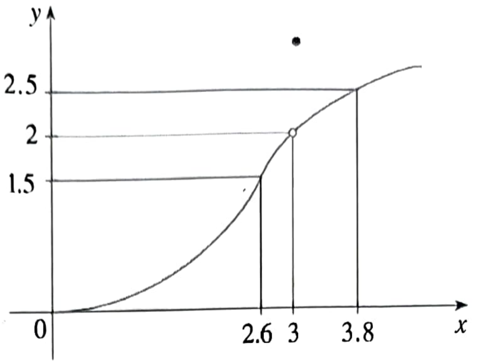
$\Rightarrow \left|\frac{(x+h)^2 - x^2}{h} - 2x\right| < \varepsilon$

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

$|x| < \varepsilon^2$

$\Rightarrow |\sqrt{x} - 0| < \varepsilon$

4. Suppose the graph of $f(x)$ is given by the following. Find a number $\delta > 0$ such that the inequality $|f(x) - 2| < 0.5$ is true whenever $0 < |x - 3| < \delta$.



$\delta = 0.4$

5. Explain the statement

$$\lim_{x \to 0+} \ln(x) = -\infty$$

in rigorous language, then prove that it's true.

~~First, $e^{\ln(x)} = x$. We really need the continuity of e^x here. Then,~~

~~$e^{\lim_{y\to 0} \ln(x)} \to 0$, but $e^x > 0$ for all x and we must have so $e^y < \delta$ for $y <$~~

$\lim_{x \to 0} \ln(x) = \lim_{y \to \infty} \ln(1/y) = \lim_{y \to \infty} -\ln(y)$;

6. Prove that the limit

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

does not exists.

$\sin\left(\frac{1}{2k\pi + \pi/2}\right) = 1$

$\sin\left(\frac{1}{2\pi k + \frac{3\pi}{2}}\right) = -1$

For $|x| < \delta$, $\exists$ $\frac{1}{2\pi k + \pi/2}, \frac{1}{2\pi k + 3\pi/2} \in (-\delta, \delta)$

So $|\sin(1/x) - a| > 2$ for some x in this interval, any a

7. Suppose $f(x, y)$ is a function that takes two real numbers as inputs and gives one real number as output. How should we define the expression

$$\lim_{(x,y) \to (0,0)} f(x, y) = L$$

for some number L? (Hint: the distance between two points (x_1, y_1) and (x_2, y_2) on the plane is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.)

$\forall \varepsilon\ \exists \delta$ $\sqrt{(x_1)^2 + (x_2)^2} < \delta \Rightarrow |f(x) - L| < \varepsilon$

$(\varepsilon > 0)$